THE PROBLEM OF SECURING TEACHERS OF COLLEGIATE MATHEMATICS FOR WARTIME NEEDS

The eighteen-year-old draft law and the inauguration of the proposed Army and Navy programs for colleges and universities are creating serious personnel problems for departments of mathematics. In an attempt to alleviate this situation, partially at least, the American Mathematical Society and the Mathematical Association of America are establishing in the Philadelphia Office of the Society a bureau to provide information regarding available teachers of collegiate mathematics. The cooperation of all mathematicians is needed in order to make this plan a success.

The government has indicated that approximately 250,000 trainees will be sent to a selected group of about 300 colleges and universities. It is estimated that the full-time services of at least 2500 teachers of mathematics will be required for this program. In addition, institutions will necessarily provide a normal program for men under 18, for women, who are electing mathematics in increasing numbers, and for students rejected, for any reason, by Selective Service. Institutions will also continue special contracts already in existence such as meteorology and aviation pre-flight schools.

To provide for all this instruction there are available about 3,000 persons now functioning as teachers of mathematics in four-year colleges. To ensure the proper handling of this instruction, readjustments of and additions to teaching staffs of many institutions will be necessary. For some institutions, additional personnel might be secured from the following sources:

(1) Members of departments of mathematics in institutions which do not assume government programs;

(2) Persons now teaching in non-critical fields who have had sufficient training in mathematics to make them valuable in this field. These persons should be recruited for the teaching of mathematics rather than being allowed to assume non-academic employment. This should probably be the main source of supply.

It is in order to promote an orderly solution of these problems that the American Mathematical Society and the Mathematical Association of America have established the bureau of information regarding teachers of collegiate mathematics. The purpose is two-fold:

(1) To collect information concerning persons who are trained in mathematics and who are or will be available for teaching during the emergency at institutions other than their own. It is requested that

the names and present addresses of persons who are available for employment as teachers of collegiate mathematics be reported to the address given below. Additional information concerning qualifications will be secured directly from the mathematicians so reported.

(2) To furnish to chairmen of departments of mathematics who need additional personnel the information obtained through (1). As the employment offered will, in most cases, be temporary, an attempt will be made to furnish the chairman with an adequate list, mainly of persons living in his vicinity. *The purpose of the bureau is to furnish only information and not recommendations.* The department chairman will, of course, investigate on his own responsibility the qualifications of the available teachers.

The country's awakened consciousness of the need for mathematics as a preparation for service in the armed forces imposes on the mathematical profession the grave responsibility to meet this need by providing the best possible teaching. The Society and Association aim to make a definite contribution to the war effort by supplying information regarding available teachers of mathematics. To do this, the full cooperation of all mathematicians is needed. It is impossible to predict what portion of the demand will be met in this manner. It is likely that teachers will become available less rapidly than the demands for additional personnel are made, but the cooperation of mathematicians will enable the bureau to function effectively.

All correspondence in this connection should be addressed to:

Committee on Available Teachers
of Collegiate Mathematics
110 Bennett Hall
University of Pennsylvania
Philadelphia, Pa.

Committee:
W. D. CAIRNS
ARNOLD DRESDEN
J. R. KLINE

March, 1943

OCCUPATIONAL CLASSIFICATION OF MATHEMATICIANS

MEMORANDUM F

Occupational Bulletins No. 10 and No. 23 have recently been amended by Selective Service Headquarters. These are the directives upon which chairmen of departments of mathematics have been basing requests for the occupational classification of members of their staffs and of students majoring in mathematics.

It is to be noted particularly that, according to both Bulletins, a graduate student in mathematics may be considered for occupational classification only if, in addition to pursuing graduate studies, he is acting as a graduate assistant and is engaged in:

(a) classroom instruction for not less than twelve hours per week; or

(b) scientific research certified by a recognized federal agency as related to the war effort.

The amended Bulletins supersede the original directives. In making requests for occupational classification, chairmen of departments of mathematics are advised to attach copies of the amended Bulletins to Form 42A. Copies of the revised Bulletins will be furnished, upon request, by the office of the Secretary. In all other details, the procedure recommended in Memorandum E, distributed on October 28, 1942, and published in the November, 1942, Bulletin of the American Mathematical Society, remains unchanged.

J. R. KLINE,
Secretary

January 30, 1943

ON THE ZEROS OF THE DERIVATIVES OF A FUNCTION AND ITS ANALYTIC CHARACTER

G. POLYA

1. **Introduction.** How do the zeros of the nth derivative $f^{(n)}(x)$ behave when n becomes very large? How does this behavior depend on the analytic nature of the function $f(x)$? At first sight this question appears to be a little out of the way. In fact it is intimately connected with essential parts of the theory of functions. In the Hadamard theory of the singularities of the Taylor series we consider the sequence of the derivatives $f(x), f'(x), f''(x), \cdots$ for a fixed complex value of x; in the theory of quasi-analytic functions we consider the same sequence for variable real x, and we consider especially the maximum absolute value of each of its terms in a given interval; now we consider again the same sequence $f(x), f'(x), f''(x), \cdots$ for variable x, that may be complex or real, and we consider especially the values of x for which its terms vanish. In all three cases the main object of consideration is the connection of the chosen feature of the sequence $f(x), f'(x), f''(x), \cdots$ with the analytic nature of the function $f(x)$.

There is a special reason for giving a general survey of results about the zeros of successive derivatives just now. The subject received quite recently essential contributions from several mathematicians in this country, from R. P. Boas, Einar Hille, A. C. Schaeffer, I. J. Schoenberg, Gabor Szegö, J. D. Tamarkin, D. V. Widder, Norbert Wiener and the present speaker. All these contributions are linked together and seem to open the door to further results and to a well connected harmonious theory.

In this short talk I should like to emphasize the main outline and the essential divisions of the theory in so far as they are recognizable today. In order to do this as clearly and intuitively as I can, I have to discuss some of the older results, to leave out some of the newer ones and to sacrifice precise details except for a few central points.

Our subject has two main parts. In one we consider analytic functions of a complex variable, in the other real-valued functions of a real variable. I proceed to describe these two parts in this order.

An address presented before the New York meeting of the Society on April 3, 1942, by invitation of the Program Committee; received by the editors May 8, 1942.

THE COMPLEX DOMAIN

2. **Statement of the problem.** In the next sections (2 to 7) $f(z)$ denotes an analytic function of the complex variable z. Let us assume that $f(z)$ is single-valued and analytic in the whole plane, except maybe in certain isolated singular points.

Let us consider the zeros of $f(z)$; they form a more or less irregular distribution of points in the plane. Now let us pass to the derivative $f'(z)$ and consider its zeros; they are in general different from those of $f(z)$ and form a new distribution of points. If we pass, by differentiating once more, to $f''(z)$ and its zeros, the picture changes again, the points appear to have moved into a third position. Corresponding to the sequence of the derivatives $f(z), f'(z), f''(z), \cdots$ we have a sequence of distributions of points formed by the zeros of the successive derivatives. Can we discover in this sequence of point-distributions some definite trend?

It is not bad to start from a vague question that is intuitive and suggestive, provided that it leads us finally to some precise question. What precise question is suggested by the intuitive process we are considering, the migration of zeros in the course of successive differentiations?

In fact, not only one question, but several different precise questions can be put. But we need a few definitions.

(I) Let us consider any fixed regular point of $f(z)$; let us assume, to be concrete, that $z=0$ is a regular point. Let r_n denote the absolute value of the zero of $f^{(n)}(z)$ which is nearest to the origin.

(II) Let us consider any fixed closed circle in which $f(z)$ is regular. Let us assume that $f(z)$ is regular in the closed unit circle $|z| \leqq 1$, and let N_n denote the number of zeros of $f^{(n)}(z)$ (with multiplicity) in this circle.

(III) We define a set D', attached to the function $f(z)$, in the following way. Consider a point z of the plane. If there exists a circle of center z in which only a finite number of derivatives vanishes, z does not belong to D'. In the other case, when any (arbitrarily small!) fixed circle with center z contains the zeros of an infinity of derivatives $f^{(n_1)}(z), f^{(n_2)}(z), \cdots$, z belongs to D'. The set D' is in some sense the final position of the zeros of $f^{(n)}(z)$; let us call D' the *final* set of $f(z)$. If $f(z)$ is meromorphic or entire, the final set D' can also be described as the derived set of the denumerable set D of all zeros of all derivatives $f(z), f'(z), f''(z), \cdots$ (provided that points in which an infinity of derivatives vanishes are also counted as points of D').

How do the zeros of the derivatives approach a regular point? What can we say about the sequence $r_1, r_2, r_3, \cdots$? Do the zeros of

$f^{(n)}(z)$ become denser as n increases? What about the sequence $N_1, N_2, N_3, \cdots$? Does the configuration of the zeros of $f^{(n)}(z)$ tend to a final position? Can we find the final set D'?

3. **Meromorphic functions.** The answer to all these questions depends on the analytic character of the function $f(z)$. For functions of certain sorts we can discover a trend, and this trend is particularly easy to recognize when $f(z)$ is a meromorphic function. In this case the trend may be roughly described by saying that the poles of $f(z)$ act as repulsive centers on the zeros of $f^{(n)}(z)$.

We can give a precise description after a precise definition. Those points of the plane which are nearer to a pole a of $f(z)$ than to any other pole of $f(z)$ constitute the *domain* of the pole a. If the domains of two poles, a and b, have common boundary points, these are on the perpendicular bisector of the line that joins a to b. Therefore the boundary of the domain of the pole a is a polygonal line, and the domain itself is an open convex set. The zeros of $f^{(n)}(z)$ are pushed toward the polygonal boundary of the domain of the pole as n increases. *The final set of a meromorphic function contains no point interior to the domain of any pole but contains all points which are on the common boundary of the domains of two or more poles* [20].

Observe that the final set depends only on the position of the poles of $f(z)$, but is independent of the multiplicity of these poles, or their residues, and so on.

4. **Functions with an essential singularity.** Let us consider next the case in which $f(z)$ has just one finite singular point, the point $z=0$, say. If this point is a pole the whole plane is its domain, the zeros of $f^{(n)}(z)$ are pushed towards ∞, they have no finite limiting point, the final set is empty. But if the point $z=0$ is an essential singular point, the situation is quite different; the trend of the movement of the zeros of $f^{(n)}(z)$ with increasing n is much more difficult to recognize, and it is no longer true that this trend depends only on the position of the singular point. For example, if

$$f(z) = (1/z)e^{-1/z},$$

all zeros of $f^{(n)}(z)$ are positive and real and the final set is the positive half of the real axis. If we change z into $ze^{i\alpha}$, where α is a fixed real number, $0<\alpha<2\pi$, the singularity of $f(z)$ remains at the point 0, but the zeros of $f^{(n)}(z)$ tend to condense along another ray issued from the singular point; the final set changes position.

5. **Entire functions.** Let us now consider a function with no finite

singular points, an entire function. Can we recognize the trend of the zeros of $f^{(n)}(z)$ as $n \to \infty$? We may say that this trend depends mainly on the rate at which $f(z)$ increases when $|z| \to \infty$, that is, on the order of $f(z)$. For certain entire functions of order 1, as, for example, for

$$f(z) = \sin z$$

the differentiation does not change essentially the density of the distribution of the zeros. If the order of $f(z)$ is less than 1 the differentiation tends to scatter the zeros; the zeros of $f^{(n)}(z)$ tend to move out to ∞ as n increases; their distribution becomes thinner. If the order of $f(z)$ is greater than 1 the differentiation tends to concentrate the zeros; the zeros of $f(z)$ tend to move in from ∞ as n increases; their distribution becomes denser. In short, the point at ∞ exerts on the zeros of $f^{(n)}(z)$ attraction or repulsion, according as the order of $f(z)$ is less than 1 or greater than 1.

Passing to precise statements, if λ denotes the order of the entire function $f(z)$, we have

$$\limsup_{n\to\infty} (\log n)^{-1} \log r_n \geqq (1 - \lambda)/\lambda, \tag{1}$$

$$\liminf_{n\to\infty} (\log n)^{-1} \log N_n \leqq (\lambda - 1)/\lambda. \tag{2}$$

The inequality (1) was found, at least for $\lambda < 1$, by Ålander [1]. More relations and more precise relations of this kind were discovered by Gontcharoff and several authors following his line of research.[1] The inequality (2) seems to be new. Both inequalities are "exact," that is, the sign of equality may be attained for certain special functions. Furthermore we cannot exchange lim inf and lim sup or change either into lim without falling into false assertions.[2]

6. **Real entire functions.** An entire function $f(z)$ is called *real* if it takes real values for real z, or, what is the same, if the coefficients of its Maclaurin series are all real. The set of zeros of a real entire function is symmetrical with respect to the real axis. The real axis seems to exert an influence on the complex zeros of $f^{(n)}(z)$; it seems to at-

[1] Gontcharoff [14]. See also Takeneka [30–34], Kakeya [16], Whittaker [35, 36], Schoenberg [28]. Boas [9] (see also Levinson [17]) introduced a quantity related to but different from r_n; let s_n denote the radius of the largest circle of center $z=0$ in the interior of which $f(z)$ is regular and $f^{(n-1)}(z)$ is *univalent*. Visibly $r_n \geqq s_n$. It seems that s_n has more simple properties than r_n; especially we have the following double inequality $\liminf_{n\to\infty} (\log n)^{-1} \log s_n \leqq (1-\lambda)/\lambda \leqq \limsup_{n\to\infty} (\log n)^{-1} \log s_n$.

[2] If $\lambda \leqq 1$ the final set may be empty and we may construct entire functions of any given order whose final set consists of just one point. *If $\lambda > 1$, must the final set necessarily contain a point?*

tract these zeros when the order is less than 2, and it seems to repel them when the order is greater than 2.

We may pass to exact statements, but these are, for the time being, hypothetical.

A. *If the order of the real entire function $f(z)$ is less than 2, and $f(z)$ has only a finite number of complex zeros, then its derivatives, from a certain one onwards, will have no complex zeros at all.*

B. *If the order of the real entire function $f(z)$ is greater than 2, and $f(z)$ has only a finite number of complex zeros, then the number of the complex zeros of $f^{(n)}(z)$ tends to infinity as $n \to \infty$.*

Both theorems are hypothetical, but A is better known than B. A weaker theorem than A, with 2/3 instead of 2, was first proved, then 2/3 was replaced by 1, and at last by 4/3.[3] So we know today that real entire functions of order less than 4/3 lose all their complex zeros after a finite number of differentiations provided they had only a finite number to start with. But the advance from 4/3 to 2, that is, to the full theorem A, is still to be made.

B is supported mostly by "experimental" evidence.[4] I quote two interesting examples

$$(3) \qquad e^{-z^{2k}}, \qquad e^{-e^{z}}.$$

The final set of the first entire function (3) (k is a positive integer) consists of k straight lines passing through the origin and dividing the plane into $2k$ equal angles [20]. The final set of the second entire function (3) consists of an infinity of parallel lines, dividing the plane into congruent strips of width 2π.[5] The final set of both functions contains the whole real axis, but also points outside the real axis (provided $k \geqq 2$).

7. **Another open question about real entire functions.** The consider-

[3] The hypothetical theorem A (or rather a slightly more general theorem in which certain entire functions of order 2 are also considered) was found independently by two authors, A. Wiman and G. Polya. The theorem with 2/3 was stated as proved by Polya [21, see p. 27, annotation 21]; proved, with reference to this first statement, almost simultaneously by Ålander [4] and Polya [22]. The theorem with 1 was proved by Wiman [39], that with 4/3 by Polya [23]. See also Wiman [38].

[4] A first remark in the direction of theorem B was made by Polya [18, 19]. (The role of the order 2 is indicated by the following theorem, due to Laguerre. *A real entire function that has only real zeros is or is not a limit of polynomials that have only real zeros according as its order is less than 2 or greater than 2.*) A more precise hypothetical theorem was formulated by A. Wiman and investigated by Ålander [1, 2] who used a geometrical method.

[5] Suggested by the present speaker, proved by G. Szegö (unpublished).

tion of the general trend of the zeros of $f^{(n)}(z)$ and some examples, principally (3), may lead us to the following hypothetical theorem.

C. *If a real entire function of order greater than* 1 *remains bounded for real values of the variable, then its final set contains the whole real axis.*

I state this theorem without any pretension to the gift of prophecy. I state it because I think that the problem to prove or to disprove it, is neither too trivial nor completely inaccessible.

THE REAL DOMAIN

8. **Notation and introductory results.** We change our notation. Henceforth $f(x)$ denotes a real-valued function of the real variable x, that is defined and possesses derivatives of all orders in a certain interval I, and N_n denotes the number of changes of sign of $f^{(n)}(x)$ in I.

$f(x)$ need not be analytic in I. The analyticity of $f(x)$ is bound to the condition that to each point a of I corresponds a positive $\rho = \rho(a)$, such that, in the common part of the two intervals I and $(a-\rho, a+\rho)$, $f(x)$ is represented by a convergent power series in $(x-a)$. If $f(x)$ is analytic in I, its analytic continuation is completely determined by its values at the points of I and, if the continuation exists, it may turn out that $f(x)$ is meromorphic, or entire, or entire of finite order, and so on.

The number N_n may be infinite. But if the interval I is closed, and $f(x)$ is analytic in I, not only is N_n a finite number for each n but the increase of N_n with n is limited by the following theorem.

If $f(x)$ is analytic in the closed interval I, then

$$\liminf_{n\to\infty} n^{-1} N_n \text{ is finite.} \tag{4a}$$

If $f(x)$ is an entire function

$$\liminf_{n\to\infty} n^{-1} N_n = 0. \tag{4e}$$

If $f(x)$ is an entire function of the finite order λ

$$\liminf (\log n)^{-1} \log N_n \leqq (\lambda - 1)/\lambda. \tag{4f}$$

If $f(x)$ is an entire function of exponential type (that is, if its increase does not exceed the finite type of order 1)

$$\liminf_{n\to\infty} N_n \text{ is finite.} \tag{4t}$$

The letters a, e, f, t will be used systematically in distinguishing formulas, a reminding us of analytic, e of entire, f of finite order, t of

exponential type. Observe that the increase of N_n is more and more restricted as we pass along the cases a, e, f, t.

The theorem we just stated starts from a given analytic character of the function (as analytic, entire and so on) and reaches conclusions concerning the zeros of the derivatives, especially conclusions concerning the behavior of the sequence $N_1, N_2, N_3, \cdots$; the theorem seems to be new, it is easy to prove by "complex variable" methods, and appropriate to introduce our subject. We pass now to results of a different character in which we start from the derivatives, especially from properties of the sequence $N_1, N_2, N_3, \cdots$, and reach conclusions about the analytic nature of the function, using mainly "real variable" methods.

9. **All derivatives are of constant sign.** The first relationships between the analytic character of a real function and the qualitative behavior of its derivatives were discovered by S. Bernstein. One of his simplest results is concerned with the condition

$$N_n = 0, \qquad n = 0, 1, 2, \cdots, \tag{5}$$

and may be stated as follows.

If no derivative of $f(x)$ changes sign in an open interval I, then $f(x)$ is analytic in I.

This result came as a great surprise at the time of its first publication [5], and it still occupies a central position. It is the common starting point of many questions and generalizations.

10. **An infinity of derivatives is of constant sign.** A first generalization of the condition (5) is due to S. Bernstein himself [5].

If infinitely many derivatives $f^{(n_1)}, f^{(n_2)}, \cdots$ are of constant sign in the open interval I,

$$N_{n_k} = 0, \qquad k = 1, 2, 3, \cdots, \tag{6}$$

and if the sequence $n_1, n_2, \cdots$ does not increase more rapidly than a geometric progression, that is, if there is a fixed quantity Q such that

$$n_{k+1}/n_k < Q, \qquad k = 1, 2, 3, \cdots, \tag{7}$$

then it still may be asserted that $f(x)$ is analytic in I.

Recently R. P. Boas gave a new proof of this result and showed by interesting examples that the condition expressed by (6) and (7) cannot be essentially relaxed without endangering the truth of the theorem [10].

11. **All derivatives are of constant sign. Influence of the signs.**

Let us return to the central condition (5). If it holds, each derivative of $f(x)$ has a constant sign in the open interval I, and, therefore, each derivative is monotonic in I, and so is the absolute value of each derivative. In fact, since

$$(d/dx)\left[f^{(n)}(x)\right]^2 = 2f^{(n)}(x)f^{(n+1)}(x)$$

the absolute value of $f^{(n)}$ is steadily increasing or steadily decreasing, according as $f^{(n)}$ and $f^{(n+1)}$ are or are not of the same sign. We say that $f^{(m)}$ and $f^{(n)}$, where $m<n$, belong to the same *block* if and only if the absolute values of $f^{(m)}, f^{(m+1)}, \cdots, f^{(n-1)}, f^{(n)}$ all vary in the same sense, that is, all increase or all decrease. Thus $f^{(n)}$ and $f^{(n+1)}$ belong to different blocks if and only if

$$f^{(n)}(x)f^{(n+2)}(x) < 0$$

in I. Let $l_1, l_2, l_3, \cdots$ denote the lengths of the successive blocks into which the sequence $f, f', f'', \cdots$ is decomposed; we assume here that no block has infinite length, and that therefore there is an infinity of blocks. S. Bernstein discovered that the lengths of these blocks are connected with the analytic nature of f. He proved the following theorem.[6]

If, in the open interval I,

$$l_n = o(l_1 + l_2 + \cdots + l_n) \tag{7e}$$

$f(x)$ is an entire function. If there exists a positive number λ, $\lambda>1$, such that

$$l_n = O\left([l_1 + l_2 + \cdots + l_n]^{(\lambda-1)/\lambda}\right) \tag{7f}$$

$f(x)$ is an entire function of finite order not exceeding λ. If

$$l_n = O(1) \tag{7t}$$

$f(x)$ is an entire function of exponential type.

12. **An infinity of derivatives is of constant sign. Influence of the signs.** D. V. Widder found recently [37] that $f(x)$ *is necessarily an entire function of exponential type if, in I,*

$$(-1)^m f^{(2m)}(x) \geqq 0, \qquad m = 0, 1, 2, \cdots. \tag{8}$$

This condition would imply, *if* we knew that no derivative of $f(x)$ vanishes in I, that $f^{(2m)}$ and $f^{(2m+1)}$ belong to different blocks, and

[6] Bernstein's statement of his results [7, 8] is different in expression but not in substance from the statement given here.

that, therefore, no block has a length greater than two, $l_k \leqq 2$. But, in fact, Widder's condition (8) does not say anything about the non-vanishing of the derivatives of odd order in I, and, therefore, Widder's theorem is not contained in the third part of Bernstein's theorem stated in the foregoing section; (8), although of much more special nature, does not imply (7t).

R. P. Boas and the present speaker set out to find a common generalization of S. Bernstein's result, stated in the foregoing section, and of Widder's result, quoted in the present section. They obtained the following theorem [12, 13].

Let $\{n_k\}$ and $\{q_k\}$ be sequences of positive integers, $\{n_k\}$ strictly increasing. Assume that $f^{(n_k)}$ and $f^{(n_k+2q_k)}$ do not change sign in I,

$$N_{n_k} = N_{n_k+2q_k} = 0, \qquad k = 1, 2, 3, \cdots, \tag{9}$$

that

$$f^{(n_k)}(x) f^{(n_k+2q_k)}(x) \leqq 0$$

in I, and that

$$q_1 + q_2 + \cdots + q_k = O(n_k).$$

If

$$n_k - n_{k-1} = o(n_k), \qquad q_k = o(n_k), \tag{10e}$$

then $f(x)$ is an entire function. If there is a positive number λ, $\lambda > 1$, such that

$$n_k - n_{k-1} = O(n_k^{(\lambda-1)/\lambda}), \qquad q_k = O(n_k^{(\lambda-1)/\lambda}), \tag{10f}$$

then $f(x)$ is an entire function of finite order not exceeding λ. If

$$n_k - n_{k-1} = O(1), \qquad q_k = O(1), \tag{10t}$$

then $f(x)$ is an entire function of exponential type.

This theorem is more general than that of Bernstein quoted in the foregoing section, since the condition (9) is less restrictive than (5). We derive Bernstein's theorem from the present one by putting $l_1 + l_2 + \cdots + l_k = n_k$ and $q_k = 1$, for $k = 1, 2, 3, \cdots$. We obtain Widder's result, quoted in this section, by putting $n_k = 2k$ and $q_k = 1$.

13. **No derivative changes sign more than a given number of times. The case of periodic functions.** Let us return again to the fundamental theorem considered in §9. S. Bernstein derived the analyticity of the function from the condition (5). Is it possible to derive analyticity from the less restrictive condition

$$N_n = O(1), \tag{11}$$

that is, does the condition that no derivative changes sign more than a fixed number of times imply the analyticity of the function? This question was first answered in the special case of periodic functions by Norbert Wiener and the present speaker [25], and led to the following precise result.

If no derivative of the periodic function $f(x)$ of period 2π has more than $2m$ changes of sign in a period, $f(x)$ is a trigonometric polynomial of order not exceeding m,

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_m \cos mx + b_m \sin mx$$

with certain constants $a_0, a_1, b_1, \cdots, a_m, b_m$.

Any derivative of a trigonometric polynomial of order m is a trigonometric polynomial of the same order and has no more than $2m$ real zeros in a period. But if a periodic function is not a trigonometric polynomial the number of changes of sign of the nth derivative must tend to infinity with n; this is contained in the theorem. We can restate the theorem by saying that the number of changes of sign of the nth derivative of a periodic function does or does not tend to infinity, according as its Fourier series has or has not an infinity of coefficients different from zero. Thus the theorem discloses a new characteristic property of trigonometric polynomials, the uniform boundedness of the changes of sign of all derivatives.

14. **The number of changes of sign of the nth derivative has a prescribed bound depending on n. The case of periodic functions.** The theorem of the preceding section became a starting point of new questions and new generalizations. A first generalization may be formulated as the following tripartite theorem.

Let $f(x)$ denote a periodic function and N_n the number of changes of sign of $f^{(n)}(x)$ in a period. If

$$N_n < 2n/\log n \tag{11e}$$

for sufficiently large n, then $f(x)$ is an entire function. If there is a λ $(\lambda > 1)$ such that

$$\limsup (\log n)^{-1} \log N_n \leqq (\lambda - 1)/\lambda, \tag{11f}$$

$f(x)$ is an entire function of finite order not exceeding λ. If

$$N_n = O(1) \tag{11t}$$

$f(x)$ is an entire function of exponential type.

A similar but less precise tripartite statement was obtained by Norbert Wiener and the present speaker [25]. The exact form just

stated is due to Gabor Szegö [29]. As it may be shown by suitable examples, the condition (11f) is completely sharp, and (11e) differs very little (if it differs at all) from a completely sharp condition.

The condition (11t) reproduces (11), and the third part of the theorem is identical with the theorem of the foregoing section; in fact, as is well known, a periodic entire function of exponential type is just a trigonometric polynomial. The conditions (11e), (11f), (11t) are parallel to the foregoing series of conditions (4e), (4f), (4t); (7e), (7f), (7t); and (10e), (10f), (10t). The hypotheses (11e), (11f), (11t) restrict more and more the increase of N_n; the last one does not allow any "increase" at all we may say. The corresponding conclusions (that $f(x)$ is entire, entire of finite order, entire of exponential type) may be expressed by a more and more rapid decrease of the coefficients of the Fourier series; in the last case the decrease is so abrupt that all coefficients are equal to 0 from a certain one onward.

15. **Other generalizations.** The theorem considered in §13 dealt with periodic functions, and therefore with Fourier series.

It is natural to pass from periodic functions to almost periodic functions and from Fourier series to Fourier integrals (trigonometric integrals); in both cases the number of changes of sign has to be replaced by (a suitable notion of) the density of changes of sign. The first extension was hinted by Norbert Wiener and the present speaker [25] the second by J. D. Tamarkin.

A different kind of generalization was effected by Einar Hille who subjected the role of the Fourier series in the proof of the theorem of §13 to a thorough analysis. The most important property of the Fourier series $\sum c_n e^{inx}$ from the point of view of that proof is the following. The operation of differentiation d/dx changes the series in a simple way, it multiplies the coefficient c_n by the simple factor in. But the differential operation $(1-x^2)(d^2/dx^2)-2xd/dx$ acts on the development into Legendre polynomials in a similar way. This observation leads to the following theorem, one of numerous analogous results discovered by Hille [15].

Let N_n^* denote the number of changes of sign of

$$[(1-x^2)(d^2/dx^2)-2xd/dx]^n f(x)$$

in the interval $-1 \leqq x \leqq 1$. If $N_n^* = O(1)$, the function $f(x)$ possesses a development into Legendre polynomials having only a finite number of non-vanishing coefficients, that is, $f(x)$ itself is a polynomial.

16. **No derivative changes sign more than a given number of times. The general case.** The question raised at the beginning of §13 was

completely answered by A. C. Schaeffer who proved the following theorem.[7]

If, in an open interval I, no derivative of $f(x)$ changes sign more than a fixed number of times, $f(x)$ is analytic in I.

This theorem contains that of S. Bernstein, discussed in §9, since its condition (11) is less restrictive than condition (5). Schaeffer's proof is also very remarkable. It uses tools of the same general character as the proofs of the theorems discussed in §§9–12, namely inequalities between derivatives, but it adds to those known before an especially original inequality of this kind, another application of which yields a new proof of the theorem of §13, and seems to open a new vista on the hypothetical theorem C of §7.

Looking back at the new results and new analogies that have been added quite recently to our knowledge of the subject, we may obtain the impression of grasping the outlines of a well balanced theory leading to further results.

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SMITH COLLEGE

THE NOVEMBER MEETING IN SOUTH BEND

The three hundred ninety-first meeting of the American Mathematical Society was held at the University of Notre Dame, South Bend, Indiana, on Friday and Saturday, November 27–28, 1942 in connection with the Centennial Celebration of the University of Notre Dame. The attendance was about seventy including the following fifty-one members of the Society:

Leonidas Alaoglu, E. S. Allen, P. H. Anderson, W. L. Ayres, Henry Blumberg, L. M. Blumenthal, I. W. Burr, R. V. Churchill, C. E. Clark, A. H. Copeland, C. C. Craig, R. P. Eddy, Samuel Eilenberg, G. M. Ewing, N. J. Fine, C. H. Fischer, J. H. Giese, Michael Golomb, Cornelius Gouwens, L. M. Graves, G. E. Hay, M. H. Heins, T. H. Hildebrandt, A. S. Householder, M. H. Ingraham, J. H. Kenna, Joseph Landin, H. W. Linscheid, Eugene Lukacs, G. W. Mackey, Karl Menger, A. N. Milgram, C. J. Nesbitt, Jerzy Neyman, Ivan Niven, C. D. Olds, P. M. Pepper, Maxwell Reade, G. E. Schweigert, A. H. Smith, W. S. Snyder, C. E. Springer, A. H. Stone, D. M. Stone, R. M. Thrall, Bernard Vinograde, Abraham Wald, M. S. Webster, G. W. Whitehead, A. L. Whiteman, J. W. T. Youngs.

Sessions for the reading of contributed papers were held on Friday afternoon with Professor L. M. Blumenthal presiding and on Saturday morning with Professor R. V. Churchill presiding. Jointly with this meeting the University of Notre Dame held its annual Mathematical Symposium, the subject being *Modern statistics*. On Friday afternoon prior to the Society session Professor Jerzy Neyman of the University of California spoke on *Theory of confidence intervals* and at the close of the Society session Professor Abraham Wald of Columbia University spoke on *Outline of a general theory of statistical inference*. On Saturday morning before the session of the Society there were two more addresses of the Symposium: *Asymptotic properties of the likelihood ratio test* by Professor Wald and *On a class of tests equivalent in the limit to the likelihood ratio tests* by Professor Neyman. The presiding officers of the Symposium sessions were Professor C. C. Craig, Dean M. H. Ingraham, Professors Sewall Wright and A. H. Copeland. The Symposium addresses contributed greatly to the interest of the Society meetings. On Friday evening those attending the meetings were the guests of the University of Notre Dame at a buffet supper.

Titles and cross references to the abstracts of the papers read follow below. Papers 1 to 6 were read Friday afternoon, papers 7 to 12 on Saturday morning, and papers 13 to 20, whose abstract numbers are followed by the letter *t*, were read by title. Dr. G. S. Young was introduced by Professor W. L. Ayres. Paper 8 was read by Professor Nesbitt, 9 by Mr. Reed and 11 by Professor Smith.

1. Maxwell Reade: *Remarks on a paper of Beckenbach.* (Abstract 49-1-40.)

2. P. M. Pepper: *A new method for imbedding theorems.* (Abstract 49-1-76.)

3. A. N. Milgram: *A topologically invariant metric property of simple closed curves.* (Abstract 49-1-98.)

4. L. M. Graves: *Metrisation of weak convergence in Banach spaces.* (Abstract 49-1-96.)

5. G. S. Young: *Sets of axioms for the plane.* Preliminary report. (Abstract 49-1-102.)

6. Karl Menger: *Statistical generalizations of metric geometry.* (Abstract 49-1-75.)

7. Ivan Niven: *An unsolved case of the Waring problem.* (Abstract 49-1-10.)

8. C. J. Nesbitt and W. M. Scott: *Some remarks on algebras over an algebraically closed field.* (Abstract 49-1-9.)

9. Karl Menger and S. G. Reed: *On a surface not intersecting the set* R_3^1. (Abstract 49-1-97.)

10. G. M. Ewing: *Minimizing an integral on a class of continuous curves.* (Abstract 49-1-30.)

11. H. F. S. Jonah and A. H. Smith: *Zero order summability of the series conjugate to the derived Fourier series.* (Abstract 49-1-35.)

12. C. E. Springer: *Dual goedesics on a surface.* (Abstract 49-1-80.)

13. M. A. Basoco: *On the Fourier developments of a certain class of theta quotients.* (Abstract 48-11-310-*t*.)

14. Raphael Salem: *On some singular monotonic functions which are strictly increasing.* (Abstract 49-1-41-*t*.)

15. Ivan Niven: *The Pell equation in quadratic fields.* (Abstract 49-1-11-*t*.)

16. Maxwell Reade: *On a theorem of Fédoroff and Binney.* (Abstract 49-1-39-*t*.)

17. J. E. Wilkins: *A class of functions in the calculus of variations for multiple integrals in parametric form.* (Abstract 49-1-52-*t*.)

18. Brockway McMillan: *Networks of mechanisms.* Preliminary report. (Abstract 49-1-68-*t*.)

19. Max Coral: *Solution of quasi-linear partial differential equations through a characteristic initial curve.* (Abstract 49-1-29-*t*.)

20. R. L. Swain: *Approximate isometries in bounded spaces.* (Abstract 49-1-100-*t*.)

W. L. Ayres,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The three hundred ninety-second meeting of the American Mathematical Society was held at the University of California at Los Angeles on Saturday, November 28, 1942. The attendance was about sixty, including the following thirty members of the Society:

O. W. Albert, Harry Bateman, Clifford Bell, L. E. Bell, Myrtie Collier, P. H. Daus, G. C. Evans, A. I. Forsythe, G. E. Forsythe, R. D. Gordon, P. G. Hoel, G. H. Hunt, D. H. Hyers, C. G. P. Kuschke, C. C. Lin, W. E. Mason, A. D. Michal, Knox Millsaps, W. T. Puckett, G. E. F. Sherwood, S. C. Snowdon, A. E. Taylor, B. P. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, L. F. Walton, W. M. Whyburn, E. R. Worthington, Max Zorn.

The meeting opened in the morning with a general session for contributed papers at which Professor P. H. Daus presided. By invitation of the Program Committee, Professor Harry Bateman of the California Institute of Technology delivered an hour address on *Some simple differential-difference equations and their related functions.* Professor G. C. Evans presided at this lecture.

The afternoon session was devoted to a Symposium on Applied Mathematics, Professor Theodore von Kármán presiding. Professor P. S. Epstein spoke on *Elastic oscillations in plates and shells*, Professor L. M. K. Boelter on *Problems in heat transfer*, and Professor J. Holmboe on *Some aspects of the circulation of viscous fluids.*

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 2 and 7 were read at the morning session; those numbered 1 and 3 to 6 were read by title. Paper 7 was read by Professor Zorn.

1. A. C. Schaeffer: *On the oscillation of differential transforms.* III. (Abstract 48-3-119-*t*.)

2. P. G. Hoel: *On indices of dispersion.* (Abstract 49-1-88.)

3. Glenn James: *On equivalence of methods of summing divergent series.* (Abstract 49-1-34-*t*.)

4. J. E. Wilkins: *A note on the Weierstrass condition for multiple integrals in the calculus of variations.* (Abstract 49-1-53-*t*.)

5. Knox Millsaps: *Characterization of the abstract exponential function.* (Abstract 49-1-36-*t*.)

6. E. J. Purcell: *Flat space congruences of order one in* [*n*]. (Abstract 49-1-77-*t*.)

7. Einar Hille and Max Zorn: *Open additive semi-groups of complex numbers.* (Abstract 49-1-32.)

A. D. MICHAL,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

In compliance with the request of the Office of Defense Transportation, the scientific program of the forty-ninth Annual Meeting of the American Mathematical Society, which was scheduled for December 28–30, 1942, in New York City, was cancelled. Therefore the Annual Meeting consisted of the meetings of the Council and the Board of Trustees which were held on December 27 and 28, 1942, in New York City, and at which the following members of the Society were present:

C. R. Adams, W. D. Cairns, W. B. Carver, Arnold Dresden, Nelson Dunford, Tomlinson Fort, R. M. Foster, B. P. Gill, Einar Hille, T. R. Hollcroft, E. M. Hull, J. R. Kline, Solomon Lefschetz, W. R. Longley, C. C. MacDuffee, Saunders MacLane, Marston Morse, G. W. Mullins, F. D. Murnaghan, R. G. D. Richardson, J. F. Ritt, J. D. Tamarkin, A. W. Tucker, Warren Weaver, S. S. Wilks, Oscar Zariski.

At the request of the authors, some of the papers submitted for presentation at the Annual Meeting were listed for presentation in person at future meetings of the Society in New York and Chicago. All other papers which had been submitted were read by title at the meeting of the Council; titles and cross references to the abstracts of these papers appear at the end of this report. At the meeting of the Council the Secretary reported that the Gibbs Lecture, scheduled to be delivered by Professor John von Neumann at the 1942 Annual Meeting, has been cancelled indefinitely since, because of war duties, the lecturer is not available.

The Board of Trustees met at 2:00 P.M. on December 27 in the Faculty Club of Columbia University and adjourned to reconvene at 1:00 P.M. on December 28. The meeting of the Council was held at 2:00 P.M. on December 27 in the Faculty Club of Columbia University.

The Secretary announced the election of the following twenty-two persons to membership in the Society:

Professor Albert Vinicio Baez, Wagner College, Staten Island, N.Y.;
Miss Helen P. Beard, Newcomb College, Tulane University;
Mr. Edward Griffith Begle, Yale University;
Dr. Isaiah Benjamin, Montreal, Quebec, Canada;
Mr. Albert Hosmer Bowker, Massachusetts Institute of Technology;
Mr. Paul Brock, Princeton University;
Professor James A. Cooley, University of Tennessee;
Mr. Charles Laurie Dolph, Princeton University;
Mr. Robert Dean Gordon, Douglas Aircraft Company, El Segundo, Calif.;
Dean Richard Wilson Horn, Dayton Y.M.C.A. College;
Mr. Bertram Kelsey Hovey, University of Pittsburgh;
Mr. Leroy M. Kelly, U. S. Coast Guard Academy;

Dr. Richard Milton Martin, Princeton University;
Dr. Anne Frances O'Neill, Smith College;
Mr. Charles H. Papas, Aircraft Radio Section, U. S. Navy Department, Washington, D.C.;
Mr. Sidney G. Reed, Jr., University of Notre Dame;
Mr. Edward Rosenthall, McGill University;
Mr. Robert Harris Scanlan, Fleetwings, Inc., Bristol, Pa.;
Mr. Peter Treuenfels, Brooklyn, N.Y.;
Mr. John Michael Walsh, University of Notre Dame;
Miss Rae Wyland, Eastman Kodak Company, Rochester, N.Y.;
Mr. Bertram Yood, Bayonne, N.J.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Brown University: Dr. Lipman Bers, Messrs. Wilfred Macdonald Kincaid and Elmer Beaumont Tolsted.
University of California at Los Angeles: Messrs. Marshall Elder and Melvin Samuel Henry.
University of Chicago: Mr. Alexander Robert Jacoby, Miss Janet MacDonald, Mr. William S. Massey, Miss Mary Kathryn Toft, Mr. Daniel Zelinsky.
University of Cincinnati: Mr. David Hal Lipsich.
College of the City of New York: Mr. Herman Chernoff.
Columbia University: Messrs. Frederick Bagemihl, Bernard Russell Gelbaum, Pierre Gutmann, Miss Agnes P. Holló, Miss Louise Wilhelmina Miller, Mr. F. Steinhardt.
Duke University: Miss Eva Ann Pirkle.
University of Illinois: Miss Mary Rosalie Bear, Dr. Theodore Bedrick, Messrs. Harry Edward Cowan, Harman Leon Harter, and Ray Gartner Langebartel, Miss Jewell E. Schubert.
Indiana University: Mr. Evar Dare Nering, Miss Valentina Annette Potor.
Iowa State College: Mr. Theodore Alfonso Bancroft.
State University of Iowa: Miss Winifred Alice Asprey.
Massachusetts Institute of Technology: Messrs. Julian Himely Bigelow, Donald Laurence Thomsen, Jr., and Edward Joseph Zadina.
University of Michigan: Messrs. Lynn Ulfred Albers, Carl Allen Bennett, Leonard Gustave Johnson, and Clarence Francis Stephens.
University of Pennsylvania: Mr. Samuel Isadore Askovitz, Mrs. Ruth Eileen Goodman.
Smith College: Miss Kathleen Ethelwyn Butcher.
Syracuse University: Miss Lucille Kathryn Pinette.
Texas Technological College: Miss Nancy Ann Miller.
University of Virginia: Mr. Marion Kirkland Fort, Jr.
University of Washington: Mr. Worthie Lefler Doyle, Jr., Miss Frances Adelia Owen.

The following appointments by President Marston Morse were reported: as tellers for the 1942 annual election, Professor E. R. Lorch and Dr. W. C. Strodt; as Committee on Arrangements for the 1943 Annual Meeting in Cleveland, Ohio, Professors J. R. Musselman (Chairman), W. L. Ayres, M. G. Boyce, O. E. Brown, W. B. Carver,

E. H. Clarke, and C. C. Torrance; as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor R. L. Wilder (committee now consists of Professors J. R. Kline, Chairman, M. H. Stone, and R. L. Wilder); for Eastern Sectional Meetings, Professor Saunders MacLane (committee now consists of Professors T. R. Hollcroft, Chairman, Salomon Bochner, and Saunders MacLane); for Western Sectional Meetings, Professor R. E. Langer (committee now consists of Professors W. L. Ayres, Chairman, T. H. Hildebrandt, and R. E. Langer); for Far Western Sectional Meetings, Professor W. M. Whyburn (committee now consists of Professors A. D. Michal, Chairman, G. C. Evans, and W. M. Whyburn).

The Secretary reported that the ordinary membership in the Society is now 2528, including 213 nominees of institutional members and 71 life members. There are also 85 institutional members. The total attendance at all meetings in 1942 was 1010; the number of papers read was 319; there were 6 hour addresses, 20 symposium addresses, and 4 Colloquium Lectures; the number of members attending at least one meeting was 641.

At the annual election which closed on December 30, and at which 455 votes were cast, the following officers were elected:

President, Professor M. H. Stone.

Vice-President, Professor L. M. Graves.

Secretary, Professor J. R. Kline.

Associate Secretary, Professor T. R. Hollcroft.

Treasurer, Professor B. P. Gill.

Members of the Editorial Committee of the Bulletin, Professors Saunders MacLane and P. A. Smith.

Member of the Editorial Committee of the Transactions, Professor A. A. Albert.

Members of the Editorial Committee of the Colloquium Publications, Professors A. B. Coble and J. F. Ritt.

Member of the Editorial Committee of Mathematical Reviews, Professor O. E. Neugebauer.

Member of the Editorial Committee of the American Journal of Mathematics, Professor G. D. Birkhoff.

Members-at-large of the Council, Professors R. P. Agnew, E. T. Bell, Richard Courant, E. R. Hedrick, and D. H. Lehmer.

Members of the Board of Trustees, Professors W. R. Longley, Marston Morse, and G. W. Mullins, Dean R. G. D. Richardson, and Dr. Warren Weaver.

In an appendix to this report are excerpts from the Report of the

Treasurer for the fiscal year 1942 as verified by the Auditors. A copy of the complete report will be sent, on request, to any member of the Society. The Board of Trustees adopted a budget for 1943 showing authorized expenditures of $49,400. This includes an appropriation of $1,000 for Mathematical Reviews from the Marion Reilly Fund.

The Librarian reported that the Library of the Society now contains 10,235 volumes of which 7,344 are bound volumes.

The American Journal of Mathematics, which is published jointly by The Johns Hopkins University and the Society, and to which the Society contributes an annual subvention of $2,500, printed 772 pages during 1942.

Certain invitations to give hour addresses were announced: Professors R. H. Cameron and Nelson Dunford for the April, 1943, meeting in New York City; Professor A. E. Taylor for the April, 1943, meeting at Stanford University.

On recommendation of the Committee on Places of Meetings (Professor A. B. Coble, Chairman), it was voted to accept the invitation of Rutgers University to hold the 1943 Summer Meeting at that institution. It was also voted to accept the invitation of Hunter College for the April, 1943, meeting in New York City. The Secretary reported that, according to the Office of Defense Transportation, the final decision regarding scientific meetings lies with the officers of the group involved; the officers are requested to weigh the necessity of each meeting against the existing transportation situation.

The Bulletin Editorial Committee reported that Professors J. H. Curtiss and D. H. Lehmer had been appointed Assistant Editors, for a period of three years, replacing Professors J. L. Doob and Morgan Ward. It was also reported that, beginning in January, 1943, invited addresses will be printed in the green issues of the Bulletin.

The Transactions Editorial Committee reported that during the year 1942 it had published 1,192 pages, the largest figure in its history. As a consequence, the heavy backlog of papers has been reduced and the interval between submission and publication has been diminished from fourteen to six or eight months. It was also reported that Professors Richard Brauer, M. S. Knebelman, and J. J. Stoker have been appointed Associate Editors, to replace Professors A. A. Albert, J. L. Synge, and G. T. Whyburn.

On recommendation of the Council, the Board of Trustees set aside a revolving fund for the purpose of publishing the new series of monographs, Mathematical Surveys. It is expected that the first two volumes of this series will appear during the first half of 1943. These will

be *Rings* by Professor Nathan Jacobson and *The problem of moments* by Professors J. A. Shohat and J. D. Tamarkin.

As a result of discussions concerning the manpower problem confronting the mathematicians, President Stone was authorized and requested to appoint a War Policy Committee to handle all problems which concern the most effective use of mathematicians in the war effort. He was also authorized to appoint a subcommittee to study means for alleviating the serious shortage of teachers of mathematics which will confront colleges and universities upon the inauguration of the Army and Navy programs.

Titles and cross references to the abstracts of papers read at the Council meeting follow below. Mr. Edward Rosenthall was introduced by Professor Gordon Pall.

1. R. P. Agnew: *Euler transformations.* (Abstract 49-1-21-*t.*)

2. R. P. Agnew: *On sequences with vanishing even or odd differences.* (Abstract 49-1-22-*t.*)

3. R. F. Arens: *Homeomorphism groups of a space.* Preliminary report. (Abstract 49-1-89-*t.*)

4. I. A. Barnett: *A note on skew-symmetric kernels.* (Abstract 49-1-23-*t.*)

5. E. F. Beckenbach: *The stronger form of Cauchy's integral theorem.* (Abstract 49-1-24-*t.*)

6. Stefan Bergman: *A formula for the stream function of compressible fluid flow.* (Abstract 49-1-58-*t.*)

7. G. D. Birkhoff: *Measure-preserving transformations of a planar ring without planar periodic points.* (Abstract 49-1-90-*t.*)

8. D. G. Bourgin: *On a theorem of Goldstine's.* (Abstract 49-1-91-*t.*)

9. D. G. Bourgin: *Quasi norms in linear topological spaces.* (Abstract 49-1-92-*t.*)

10. D. E. Christie: *Net homotopy for compacta.* (Abstract 49-1-93-*t.*)

11. R. F. Clippinger: *General remarks about the set of products of positive powers of n-by-n matrices and the associated manifold.* Preliminary report. (Abstract 49-1-26-*t.*)

12. R. F. Clippinger: *Matrix products of matrix powers.* Preliminary report. (Abstract 49-1-27-*t.*)

13. R. F. Clippinger: *Mean value theorems for a certain linear matrix differential equation.* Preliminary report. (Abstract 49-1-28-*t.*)

14. I. S. Cohen: *Some theorems on local rings.* (Abstract 49-1-1-*t.*)

15. H. S. M. Coxeter: *The map-coloring of unorientable surfaces.* (Abstract 49-1-94-*t.*)

16. J. H. Curtiss: *A note on the theory of moment generating functions.* (Abstract 49-1-85-*t*.)

17. John DeCicco: *Conformal geometry of second order differential equations.* (Abstract 49-1-70-*t*.)

18. Jesse Douglas: *Point transformations and isothermal families of curves.* (Abstract 49-1-71-*t*.)

19. R. M. Foster: *On the average resistance of an electrical network.* (Abstract 49-1-59-*t*.)

20. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* (Abstract 49-1-60-*t*.)

21. P. R. Halmos: *On automorphisms of compact groups.* I. (Abstract 49-1-3-*t*.)

22. A. E. Heins: *Some remarks on the solution of dual integral equations.* I. (Abstract 49-1-63-*t*.)

23. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of certain integral functions.* (Abstract 49-1-33-*t*.)

24. Edward Kasner: *Trajectories in a resisting medium.* (Abstract 49-1-66-*t*.)

25. Edward Kasner and John DeCicco: *Bi-isothermal systems in pseudo-conformal geometry.* (Abstract 49-1-73-*t*.)

26. Jakob Levitzki: *On semi-nilpotent ideals.* (Abstract 49-1-6-*t*.)

27. W. H. Roever: *A new formula for the deviation in range of a projectile due to the earth's rotation.* (Abstract 49-1-69-*t*.)

28. W. H. Roever: *The axonometric method of representing the points of space on a plane.* (Abstract 49-1-78-*t*.)

29. Edward Rosenthall: *Diophantine equations in arbitrary algebraic number fields.* (Abstract 49-1-13-*t*.)

30. Edward Rosenthall: *Diophantine equations in biquadratic fields.* (Abstract 49-1-14-*t*.)

31. H. E. Salzer and Abraham Hillman: *Exact values of the first* 120 *factorials.* (Abstract 49-1-84-*t*.)

32. H. M. Schwartz: *On sequences of Stieltjes integrals.* II. (Abstract 49-1-42-*t*.)

33. I. M. Sheffer: *Note on a linear transformation of "analytic" type.* (Abstract 49-1-43-*t*.)

34. W. S. Snyder: *Non-parametric surfaces and inscribed polyhedra.* Preliminary report. (Abstract 49-1-44-*t*.)

35. Otto Szász: *On Abel and Lebesgue summability.* (Abstract 49-1-46-*t*.)

36. W. C. Taylor: *Asymptotic behavior of the Abel sums of the Laguerre expansion.* (Abstract 49-1-47-*t*.)

37. W. J. Trjitzinsky: *Singular nonlinear integral equations.* (Abstract 49-1-48-*t*.)

38. S. M. Ulam: *On equivalence of functions.* (Abstract 49-1-49-*t*.)

39. S. M. Ulam: *On the length of curves, the surface area and the isoperimetric problem under a general Minkowski metric.* Preliminary report. (Abstract 49-1-81-*t*.)

40. T. L. Wade: *Euclidean concomitants of the triangle.* (Abstract 49-1-15-*t*.)

41. T. L. Wade: *On conjugate tensors.* (Abstract 49-1-16-*t*.)

42. T. L. Wade: *On the factorization of rank tensors.* (Abstract 49-1-17-*t*.)

43. T. L. Wade and R. H. Bruck: *Types of symmetries.* (Abstract 49-1-18-*t*.)

44. A. D. Wallace: *On non-alternating transformations.* (Abstract 49-1-101-*t*.)

45. Alexander Weinstein: *On a general variational method for the determination of eigenvalues.* (Abstract 49-1-51-*t*.)

46. J. E. Wilkins: *Definitely self-conjugate adjoint integral equations.* (Abstract 49-1-54-*t*.)

47. Fumio Yagi: *On a certain Stieltjes integral equation.* (Abstract 49-1-55-*t*.)

48. Antoni Zygmund: *A property of the zeros of Legendre polynomials.* (Abstract 49-1-57-*t*.)

J. R. KLINE,
Secretary

APPENDIX

Excerpts from Report of Treasurer

December 16, 1942

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1942, in form similar to that for the previous year.

The market value on November 30, 1942, of the securities in the Pool was $12,714.63 less than book value. This is to be compared with the total amount, $7,724.46, held in accounts "Reserve for Investment Losses" and "Profit and Loss on Sale of Securities." The market value on November 30, 1942, of securities in General and Restricted Funds was $1,459.25 less than book value. Profit on the sale of securities for General and Restricted Funds amounting to $295.73 is still being carried as a reserve against losses to these funds.

Income earned by General Fund Investments during the year amounted to $1,414.34. This represents a return of $1\frac{5}{8}\%$ on amounts in savings banks, and approximately $3\frac{3}{4}\%$ on securities. Investment Income earned by the Pool amounted to $7,380.12, representing a return of approximately $4\frac{1}{2}\%$. Total Investment Income from all sources was thus $8,794.46, corresponding to a yield of approximately 4%. This income and yield may be compared with $8,864.93 and approximately $4\frac{3}{8}\%$ for the fiscal year 1941.

There appears for the first time a record of income from the Estate of Robert Henderson. The terms of Dr. Henderson's will have earlier been reported to the Trustees and to the Society. There also appears the gift of $100 from Professor R. C. Archibald for use of the library.

Respectfully submitted,

Bennington P. Gill, *Treasurer*

BALANCE SHEET

Assets

	November 30, 1942	November 30, 1941
GENERAL AND RESTRICTED FUND ASSETS:		
Cash	$ 19,177.78	$ 19,380.05
Investments	33,470.47	28,814.24
	$ 52,648.25	$ 48,194.29
ENDOWMENT AND OTHER NONEXPENDABLE FUND ASSETS (POOL):		
Cash	$ 1,409.62	$ 1,156.70
Investments	163,195.68	160,395.09
	$164,605.30	$161,551.79
TOTAL ASSETS	$217,253.55	$209,746.08

Liabilities

GENERAL FUNDS:		
Sinking Fund	$ 515.99	$ 420.09
Profit on Sales of Securities	295.73	295.73
Surplus	14,201.68	9,048.08
	$ 15,013.40	$ 9,763.90
RESTRICTED FUNDS:		
Prize Funds and other Special Funds Accumulated Income	$ 4,848.52	$ 5,613.02
Income from Henderson Estate	2,275.00	
Reilly Fund Principal	3,968.13	3,968.13
Reprinting Funds	1,329.15	788.86
Colloquium	2,805.04	5,187.14
Mathematical Reviews	16,184.69	16,924.45
International Congress	6,050.33	5,943.44
Library—R. C. Archibald Gift	100.00	
Miscellaneous	73.99	5.35
	$ 37,634.85	$ 38,430.39
	$ 52,648.25	$ 48,194.29
ENDOWMENT AND OTHER NONEXPENDABLE FUNDS (POOL):		
Endowment Fund Principal	$ 60,110.68	$ 60,110.68
Prize Funds and other Special Funds	26,935.07	26,935.07
Life Membership and Subscription Reserve	4,835.09	5,304.08
Colloquium	5,000.00	5,000.00
Mathematical Reviews	60,000.00	60,000.00
Reserve for Investment Losses	3,537.46	2,728.24
Profit on Sales of Securities	4,187.00	1,473.72
	$164,605.30	$161,551.79
TOTAL LIABILITIES	$217,253.55	$209,746.08

SUMMARY STATEMENT OF INCOME AND EXPENDITURES 1941–1942

	1942		1941	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS				
Dues—Ordinary Memberships	$ 15,681.19		$ 14,970.19	
Dues—Contributing Memberships	1,054.00		1,040.00	
Dues—Institutional Memberships	7,347.35		6,468.31	
Initiation Fees	665.00		515.25	
Investment Income	4,643.14		4,147.27	
Miscellaneous	34.73		343.15	
GENERAL DISBURSEMENTS				
Secretaries		$ 5,136.86		$ 6,091.29
Treasurer		794.36		750.00
Officers' Traveling		643.61		888.93
Library		569.95		824.50
Committee Expenses		67.92		101.04
Furniture and Fixtures				51.70
Emergency		5.71		277.29
Total	$ 29,425.41	$ 7,218.41	$ 27,484.17	$ 8,984.75
Excess of General Receipts	$ 22,207.00		$ 18,499.42	
PUBLICATION				
Bulletin	$ 1,673.13	$12,101.37	$ 1,755.47	$11,037.14
Bulletin Reprinting	467.30		227.62	
Transactions	5,787.17	9,373.59	4,816.11	8,968.10
Transactions Reprinting	274.58	201.59		
Colloquium	3,814.97	6,197.07	3,437.74	2,969.55
Mathematical Reviews	10,809.17	11,548.93	21,467.56	12,969.22
Semicentennial Publications	95.20		129.29	
American Journal		2,500.00		2,500.00
Total	$22,922.22	$41,922.55	$ 31,833.79	$38,444.01
Excess of Cost of Publication		$19,000.33		$ 6,610.22
MISCELLANEOUS				
Special Funds (including Congress)	$ 1,442.39	$ 2,100.00	$9,526.51[1]	$ 1,000.00
Profit on Sales of Securities	2,713.28		103.72	
Income from Henderson Estate	2,275.00			
Gift from R. C. Archibald	100.00			
Transactions Reprinting from Surplus		100.00		
Member-Customer Items—not allocated	73.99	103.86		
Total	$ 6,604.66	$ 2,303.86	$ 9,630.23	$ 1,000.00
Difference	$ 4,300.80		$ 8,630.23	
Net Change in Assets	$ 7,507.47		$ 20,519.43	
ASSETS BEGINNING OF YEAR	$209,745.08		$189,226.65	
ASSETS END OF YEAR	$217,253.55		$209,746.08	

[1] Includes $7,738.78 principal of gifts.

BOOK REVIEWS

Algebraic topology. By Solomon Lefschetz. (American Mathematical Society Colloquium Publications, vol. 27.) New York, American Mathematical Society, 1942. 6+389 pp. $6.00.

Since the publication of Lefschetz's *Topology* (Amer. Math. Soc. Colloquium Publications, vol. 12, 1930; referred to below as (L)) three major advances have influenced algebraic topology: the development of an abstract complex independent of the geometric simplex, the Pontrjagin duality theorem for abelian topological groups, and the method of Čech for treating the homology theory of topological spaces by systems of "nerves" each of which is an abstract complex. The results of (L), very materially added to both by incorporation of subsequent published work and by new theorems of the author's, are here completely recast and unified in terms of these new techniques. A high degree of generality is postulated from the outset. The abstract point of view with its concomitant formalism permits succinct, precise presentation of definitions and proofs. Examples are sparingly given, mostly of a simple kind, which, as they do not partake of the scope of the corresponding text, should be intelligible to an elementary student. But this is primarily a book for the mature reader, in which he can find the theorems of algebraic topology welded into a logically coherent whole.

The first chapter presents the set-theoretic considerations which will underlie both the spaces to be studied later and the algebraic machinery used to study them. Topological spaces are defined as point sets in which open sets are specified subject to the three usual axioms. Mappings (that is, continuous single-valued transformations) are next defined so that their properties may be developed along with those of the space. By using topological products of open or of closed line segments, Euclidean n-space, the n-cell and the Hilbert parallelotope are introduced. A space is called *compact* if for every covering $\{U_\alpha\}$ by open sets there exists a finite subset of $\{U_\alpha\}$ which covers the space. This is the property often called "bicompactness." The principal consequences of the separation axioms particularly for compact spaces precede a definition of *normality* and *characteristic function* and the Tychonoff imbedding theorem, which, via metric spaces, leads to the Urysohn metrization theorems. A set is a *directed system* if there is a relation $>$ between certain pairs a,b of its elements such that $c>b$ and $b>a$ implies $c>a$, and for every pair a,b of elements there is an element c such that $c>a$ and $c>b$. Using these, inverse

mapping systems of Hausdorff spaces and their limit spaces are defined. The chapter ends with a definition of *homotopy*, *deformation* and *retraction*.

Abelian topological groups occupy the next chapter. The standard decomposition of a group with a finite number of generators is given. The chapter as a whole leads up to duality theorems. The first is that of Pontrjagin-van Kampen which is stated for locally compact groups G and H one of which is the character group of the other, and is proved in case G is compact, H discrete (except that reference to the literature is given for the step $G \ni g \neq 0$ implies the existence of $h \in H$ such that $gh \neq 0$). Application is made to dual directed systems of groups. If a set Ω of continuous homomorphisms α of a group G into itself forms a field (taken with discrete topology) in the sense that $\alpha g + \alpha' g = (\alpha + \alpha')g$ and $(\alpha\alpha')g = \alpha(\alpha' g)$, then G is called a *vector space* and has *linear* topology if every neighborhood of the identity of G is a sum of subspaces. If G is finite-dimensional, linear reduces to discrete topology. When a *field character* of G is defined as a homomorphism $G \to \Omega$, the field characters form a vector space H called the *character space* of G. In case G and H have linear topology and are restricted by a condition analogous to compactness, a second duality theorem, analogous to the first, is proved for them.

In Chapter III, Lefschetz defines as a *complex* an abstract system based on the "abstract complex" of Tucker except that the use of negative dimensions here permits the association with each complex X of a symmetric dual complex X^* which carries its cohomology theory. Geometric entities like *polyhedral complexes* (made up of bounded convex regions) and *Euclidean complexes* (made up of simplexes) can then be treated as special cases. Homology and cohomology theory with general coefficient groups G (in case G is a discrete field the homology groups are vector spaces) and product theory are developed abstractly both for finite and for infinite complexes having local finiteness properties. It is shown, following Steenrod, that two finite complexes have the same homology groups over every G if they have the same groups for either of the *universal* coefficient groups: G the integers, or G the reals mod 1. If G is a field of characteristic π, the Betti numbers over G are the same as those mod π. Using the group duality of Chapter II gives (1) the duality of the pth homology and cohomology groups, and (2) the Alexander duality of the pth cohomology group of a closed X and the $(p-1)$st homology group of $Y-X$ when the pth and $(p-1)$st homology groups of Y are the identity. After discussion of *product* and *join* of complexes, subdivision is treated by means of the *chain-mapping* induced by the transformation

of one complex to another; intersections are defined as chain mappings of the product $X \times X^*$ to X or, equivalently, of $X^* \times X^*$ to X^*. In terms of them the homology ring of Gordon and Freudenthal is set up and the theory of coincidences and fixed points is abstractly given. Combinatorial manifold is defined and the previous results are applied in this special case, the result of (1) for finite manifolds being the duality theorem of Poincaré, for infinite manifolds the theorem of (L) p. 314.

The Čech homology theory is next abstracted: the role of Alexandroff "nerves" being played by a directed set of complexes called a *net*, the role of the projection onto a covering of its refinement being taken by a chain-mapping. If whenever there is a projection it is unique, the net is called a *spectrum*, so the spectrum is the direct abstraction of the Čech situation, the net being a generalization. For a fixed dimension the homology group $\mathfrak{H}$ of a net is then the limit group of the homology groups of that dimension of the complexes of the net. The duals of these complexes yield a *conet* and cohomology groups which permit the treatment of intersections. A duality theorem like (1) is immediate; one like (2) is proved when the coefficient group is compact or a field. For a spectrum the limit groups Z of the groups of cycles and F of the groups of bounding cycles of the complexes of the spectrum may be defined leading to a *projective homology group* $Z/\overline{F}$ for each dimension. (The bar is the closure operator.) If the coefficient group is compact or a field, $\mathfrak{H} \approx Z/\overline{F}$. A *web* is a collection of objects, $A, B, \cdots$ between certain pairs of which a relation $\supset$ of "inclusion" has been established such that to each A,B there is a $C \supset A,B$, and a D with $A,B \supset D$. Thus from a web two directed sets may be formed; one *direct*, $A > B$ means $A \supset B$; one *inverse*, $A > B$ means $B \supset A$. Here the elements of the web are the subcomplexes of a complex or, more generally, the subnets of a net, and by means of the directed sets, homology theories are constructed for these webs. The principal application is to an infinite complex each of whose elements has a finite set of elements on its boundary and in which each element has a *diameter*. Then if $A,B, \cdots$ are subcomplexes made up of elements of decreasing maximum diameters one of the homology theories of the resulting web abstracts the Vietoris homology theory.

Chapters VII and VIII apply the algebraic machinery to spaces, nets being used to define homology groups and prove duality and intersection theorems for topological spaces in the manner of Čech. If the space is normal it makes no difference whether the nets are composed of nerves of open or of closed sets. Similarly webs serve for the Vietoris-Lefschetz homology theory of compact metric spaces

(L) p. 322. It is shown that the Čech and Vietoris homology groups over a discrete group G are isomorphic. Special finite coverings of a topological space by closed sets whose interiors are disjunct, called *gratings*, lead to a net which is a spectrum and whose net homology theory is the same as the Kurosch homology theory by finite closed coverings. The projective theory of this spectrum is the homology theory of Alexander-Kolmogoroff. In Chapter VIII the topological space is specialized to be a polyhedral or simplicial complex K and the covering to be by barycentric stars of the derived (that is, regularly subdivided) complexes of K. A proof of the topological invariance of the algebraic homology groups of K then quickly results from the Čech theory. The manifold, intersection and fixed point theories given earlier are specialized to this case and a discussion of the singular chains which played such a large part in (L) and of continuous chains is included. Finally differentiable complexes and group manifolds are discussed. Hopf by generalizing simplicial group manifolds (group not necessarily abelian) defined a Γ-manifold. Lefschetz further generalizes to obtain a Γ-complex and proves Hopf's theorems for it: the rational homology groups and ring of a Γ-complex are isomorphic with those of a finite product of odd-dimensional spheres; and every finite product of odd-dimensional spheres is a Γ-manifold.

In Appendix A by S. Eilenberg and Saunders MacLane are proved for infinite complexes results on universal coefficient groups similar to those of Chapter III for finite complexes. The group of group extensions of a given group by another is the algebraic tool used. In Appendix B, P. A. Smith describes his application of algebraic topology to the study of the fixed points of a periodic homeomorphism T of a topological space R into itself. His technique is first worked out for R a simplicial complex and T a simplicial homeomorphism. The algebra is that of special homology groups defined by means of T. Then by the Čech method it is extended to compact spaces, particularly those having the homology groups of the n-sphere, and there yields topological results. Some unsolved problems are described at the end of this appendix.

WILLIAM W. FLEXNER

Principles of mechanics. By J. L. Synge and B. A. Griffith. New York and London, McGraw-Hill, 1942. 12+514 pp. $4.50.

In their preface the authors state that mechanics stands out as a model of clarity among all the theories of deductive science, and they have succeeded very well in support of that statement in the production of this book. The notation and the arrangement are good, and

the scope of material covered exhibits well the methods and limitations of modern mechanics.

The book is divided into two parts, the first being entitled Plane Mechanics and the second Mechanics in Space. There is, however, in Part I considerable three-dimensional notation, the policy of the authors being to include the three-dimensional generality when undue complexity does not arise.

Vector methods are used throughout Part II as the fundamental mode of approach both to theory and to problems. They are also used in Part I, although in many of the problems of plane mechanics there is not scope for the full advantage of vector notation, and scalars are used a great deal instead.

The arrangements in the two parts to a large extent parallel each other. In each case statics precedes dynamics. The force concept is taken as the primitive one, rather than mass, and the principles of statics are developed through the principle of virtual work. Then in each case there follow chapters on the methods of dynamics and their applications. In Part II these are followed by a chapter on Lagrange's equations, and one on the special theory of relativity. These last two chapters are well adapted to the purpose of providing the student with an indication of what lies beyond the methods which have been given, and of the limitations of the postulates on which the theoretical structure of Newtonian mechanics rests.

The first chapter of Part I is entitled Foundations of Mechanics. It deals with the fundamental concepts and postulates of mechanics, but to appreciate it adequately the student needs a considerable degree of experience and maturity concerning things physical. The mass idea is illustrated by an analogy, but the amount of definite physical basis suggested for the idea is meager. In fact, the emphasis of the book is not on the development of an understanding of the fundamental concepts and postulates; a working knowledge of the fundamental ideas is presented in condensed form, chiefly in this first chapter, and then the development of the logical structure of mechanics is proceeded with. That, of course, is a legitimate choice, for it makes possible the inclusion of a greater body of theory and method in a single volume of moderate size. It makes the text unsuitable for a beginners' course, but it provides a valuable body of material for more advanced work.

There are, however, some aspects of the book to which the reviewer takes exception. A vector is defined (p. 18) as a directed segment, or any physical quantity which can be represented by a directed segment. The parallelogram law is regarded as giving the mathematical sum of two vectors, irrespective of whether it gives

their equivalent when they represent physical quantities, but the theory of vectors developed is applied to physical problems. A finite rotation is explicitly classed as a vector, although it is admitted that it does not obey the parallelogram law. That forces obey the parallelogram law is regarded as an axiom, but there is no mention as to whether other specific quantities obey it. The general statement is made (p. 20) that almost all physical vectors do obey the parallelogram law but there is no justification given for the statement. This seems to be a logical weakness.

On p. 88 there is an erroneous statement, namely that "μ (the coefficient of static friction) is always less than unity." This would mean that no body or material could repose on a plane inclined to the horizontal at an angle greater than 45°, which is obviously not true.

Then in the chapters on impulsive motion there is introduced some artificial and unsatisfactory terminology. It is true that there is eminent precedent for what is done (see, for example, Lamb's *Hydrodynamics*, p. 11), but nevertheless it is open to objection. In the momentum-impulse equation

$$\Delta(m\dot{x}) = \int_{t_0}^{t_1} X dt,$$

t_1 is allowed to approach t_0 and X to become infinite, the integral remaining constant. There is the statement (p. 226) that "in this limiting case of 'an infinite force acting for an infinitesimal time' there is an instantaneous change in velocity but no change in position." Later, on the same page: "no force, however large, can produce an instantaneous change in momentum." Then: "To place our new ideas on a secure foundation, we admit the concept of an *impulsive force*." This impulsive force has a component $\hat{X}$ given by

$$\hat{X} = \lim_{t_1 \to t_0} \int_{t_0}^{t_1} X dt,$$

and then it is stated that the impulsive force causes an instantaneous change in momentum given by

$$\Delta(m\dot{x}) = \hat{X}.$$

But the calling of $\hat{X}$ an impulsive force is unsatisfactory. This quantity is not a special kind of *force*; it is a different kind of physical quantity. The loose use of terms on this topic is conducive to confusion of ideas, and is in contrast with the general quality of the book.

At the end of the book there is a well written appendix on the theory of dimensions, but it seems unfortunate that the principles of it are not more in evidence throughout the text. There is one reference in it to the principle of significant figures. The latter is referred to as the physical way of thinking, in contrast with the mathematical way. That may be, but the main object of mechanics is to develop a theoretical structure which will describe certain aspects of nature, and the principle of significant figures has broader application than the rounding off of digits in a numerical calculation. It is a principle that is of frequent use in assessing relative values.

Thus, on p. 17 the authors express a preference for the tension of a spring rather than the weight of a body as a basis for the definition of force measurement, since the former is constant while the latter is variable. They state: "If the weight of a body is measured by means of a calibrated spring at two different latitudes, the results are not the same." But they are probably referring to an ideal calibrated spring, for it is doubtful if the best calibrated spring in use in any laboratory would detect a difference. To a beginner the testing of this statement by experiment would not be very convincing. Practice and theory with regard to the gravitational and absolute measure of force are justified with clarity by the principle of significant figures.

The momentum impulse equation is best interpreted by the same principle. The time interval does not *reach* the limit zero to make the change of momentum instantaneous, and the displacement is not zero but negligible.

In spite of these defects the general organization of the book is good, in both material and presentation. The printing is well done, and is remarkably free from typographical errors.

J. W. CAMPBELL

NOTES

The editorial committee of the Bulletin has in the past adhered to an upper limit of ten printed pages on the length of papers published in the Bulletin, except in the case of invited addresses. In the future no fixed upper limit will be maintained. The Council of the Society has approved the general policy that the Bulletin should be reserved for papers of more general interest and the shorter notes, while the Transactions should have the deeper and longer research papers which would be read particularly by specialists. In some cases it is anticipated that a paper submitted to one of the Society's journals may be transferred to the other, particularly when it is of intermediate length, and publication may thereby be hastened. In view of the extreme demands for space in the mathematical journals, authors are urged to make their articles as concise as is consistent with clarity. In some cases consideration should be given to the possibility of combining several short notes into one memoir.

The National Research Council has announced a new quarterly publication, *Mathematical tables and aids to computation*, edited by Professor R. C. Archibald of Brown University. The publication is to serve as a clearing-house for information concerning mathematical tables and other aids to computation. The chief parts of each issue will normally be devoted to the following topics: articles, recent mathematical tables (critical reviews), mathematical tables—errata, unpublished mathematical tables, mechanical aids to computation, notes, queries, queries-replies. The price will be three dollars a year, one dollar a number. For subscriptions and further information address Division of Physical Sciences, The National Research Council, 2101 Constitution Avenue, Washington, D. C.

Professor O. P. Akers of Allegheny College has retired.

Associate Professor J. V. Atanasoff of Iowa State College has been promoted to a professorship of mathematical physics.

Mr. O. E. Bennett has been appointed to an assistant professorship at Washington College, Chestertown, Maryland.

Assistant Professor Herman Betz of the University of Missouri has been promoted to an associate professorship.

Professor J. H. Butchart of William Woods College, Fulton, Missouri, has been appointed to an assistant professorship at Grinnell College.

Mr. C. L. Buxton of Clarkson College of Technology has been promoted to an assistant professorship.

Associate Professor C. S. Carlson of St. Olaf College, St. Paul, Minnesota, has been promoted to a professorship.

Dr. G. P. Cowan of St. John's University has retired.

Dr. J. M. Dobbie of Northwestern University has been promoted to an assistant professorship.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Dr. J. W. Givens of Northwestern University has been promoted to an assistant professorship.

Assistant Professor H. S. Kaltenborn of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Dr. H. L. Lee of the University of Tennessee has been promoted to an assistant professorship.

Mrs. D. H. Lehmer and Mrs. C. D. Shane have been appointed to lectureships in mathematics at the University of California.

Associate Professor A. N. Lowan of Yeshiva College has been promoted to a professorship.

Professor W. A. Manning of Stanford University has retired.

Mr. E. C. Molina of the Bell Telephone Laboratories has retired.

Dr. R. H. Moorman of Tennessee Polytechnic Institute has been promoted to an assistant professorship.

Dr. J. M. H. Olmsted of the University of Minnesota has been promoted to an assistant professorship.

Assistant Professor L. D. Rodabaugh of Butler University has been appointed to a professorship at Shurtleff College, Alton, Illinois.

Assistant Professor Helen G. Russell of Wellesley College is on leave for the academic year 1942–1943 and is at the University of California.

Dr. P. C. Scott of Louisiana State University has been promoted to an assistant professorship.

Dr. Abraham Spitzbart of the University of Minnesota has been appointed to a professorship at the College of St. Thomas.

Dr. Ellen C. Stokes of the New York State College for Teachers at Albany has been appointed dean of women.

Mr. J. A. Straw of the University of Louisville has been appointed to an assistant professorship at Rose Polytechnic Institute.

Dr. S. H. Taylor has been appointed adjunct professor at the University of South Carolina.

Dr. D. L. Webb of Georgia School of Technology has been appointed to an assistant professorship at Texas Technological College.

Professor D. V. Widder has been appointed chairman of the department of mathematics at Harvard University, succeeding Professor M. H. Stone.

Associate Professor Roscoe Woods is acting head of the department of mathematics at the University of Iowa during the illness of Professor H. L. Rietz.

Dr. E. L. Welker of the University of Illinois has been made an associate.

The following appointments to instructorships are announced: University of Arkansas: Mr. R. V. Simpson; Brown University: Mr. A. F. Bartholomaz, Mr. M. E. Munroe; University of California: Mrs. R. K. Wakerling; University of California at Los Angeles: Dr. R. H. Sorgenfrey; University of Illinois: Mr. J. C. Bell; Iowa State College: Mr. J. W. Beach, Mr. F. E. Bortle, Mr. R. N. Goss; University of Kentucky: Miss Mary H. Cooper, Mr. J. C. Eaves, Miss Lydia R. Fischer; Louisiana State University: Mr. J. E. Pryor; Miami University: Miss Alberta Wolfe; Michigan College of Mining and Technology: Mr. J. C. Butler, Mr. T. R. Richards, Mr. Earl Roberts; State Teachers College, Mankato, Minnesota: Mr. C. J. Kirchen; University of Minnesota: Mr. W. D. Munro; Muskingum College, New Concord, Ohio: Mr. L. C. Knight, Jr.; University of North Carolina: Dr. I. S. Cohen; Northwestern University: Miss Louise M. Sagen; University of South Carolina: Miss Flora Dinkines; University of Southern California: Miss F. Marian Clarke, Mr. E. C. Rex, Mr. Veryl Throckmorton; Agricultural and Mechanical College of Texas: Mr. A. R. Wapple; Texas Technological College: Miss Annie N. Rowland; Tuskegee Institute: Dr. J. E. Wilkins; Williams College: Mr. E. F. Gillette; University of Wisconsin: Dr. D. T. McClay.

Dr. A. E. Mayer of London, England, died November 3, 1942. He was formerly connected with the University of Vienna.

Professor Emeritus Alfred Baker of the University of Toronto died October 27, 1942, at the age of ninety-four years. He had been a member of the Society from 1891 to 1920.

Professor W. M. Carruth of Hamilton College died January 23, 1943. He had been a member of the Society since 1907.

Professor Emeritus C. H. Currier of Brown University died January 5, 1943. He was a member of the Society from 1908 to 1939.

Professor Emeritus I. M. DeLong of the University of Colorado died September 2, 1942. He had been a member of the Society since 1891.

Professor E. L. Dodd of the University of Texas died January 9, 1943. He had been a member of the Society since 1904.

Dean Emeritus G. H. Ling of the University of Saskatchewan died October 21, 1942, at the age of sixty-eight years. He had been a member of the Society since 1894.

Assistant Professor L. I. Neikirk of the University of Washington died December 10, 1942. He had been a member of the Society since 1903.

Professor Emeritus J. H. Scarborough of Central Missouri State Teachers College died November 25, 1942. He had been a member of the Society from 1906 to 1940.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

103. A. A. Albert: *Algebras derived by non-associative matrix multiplication.*

Let $\mathfrak{C}$ be any J-involutorial algebra and define its isotopes $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$ by $x, y=x(yJ)$, $(x, y)=(xJ)y$, $[x, y]=(xJ)(yJ)$, respectively. When $\mathfrak{C}$ has a unity quantity the algebras $\mathfrak{C}_\rho$, $\mathfrak{C}_x$, $\mathfrak{C}_{x\rho}$ are all semi-simple if and only if $\mathfrak{C}$ is semi-simple, they are simple if and only if $\mathfrak{C}$ is either simple or a direct sum $\mathfrak{S}\oplus\mathfrak{S}J$ where $\mathfrak{S}$ is simple. If $\mathfrak{N}$ is the radical of $\mathfrak{C}$ then $\mathfrak{N}=\mathfrak{N}J$ forms the radical of the isotopes. The structure of row algebras, that is, subalgebras of $\mathfrak{C}_\rho$ is determined. In particular it is shown that there exist real linear spaces of matrices forming algebras under row by row but not under row by column multiplication. They are never semi-simple. (Received January 26, 1943.)

104. Joseph Bowden: *The quaternary permutation function and a generalization of Newton's binomial theorem and Vandermonde's permutation theorem.*

If a and d are any finite numbers, m any integer, r a primary number and ρ an integer whose elements are the primary numbers r_1 and r_2, define $(a, d)P(m, r)=\prod_{k=m+1}^{m+r}(a-(k-1)d)$ and $(a, d)P(m, \rho)=(a, d)P(m-r_2, r_1):(a, d)P(m-r_1, r_2)$. From these definitions it follows that $(a, d)P(m, 0)=1$ and $(a, d)P(m, \rho)=:[(a, d)P(m+\rho, -\rho)]$. The operation P is quaternary because it acts on four operands. From the quaternary permutation function $(a, d)P(m, \rho)$ by putting $d=0$ it is found that $(a, d)P(m, \rho)=a^\rho$. By putting $d=1$ or $a=\rho d$ or $m=0$ three ternary functions are obtained. By making two of these three substitutions three binary functions are obtained. In particular if $d=1$ and $m=0$, $(a, d)P(m, \rho)=aP\rho$, the binary permutation function. By making all three of these substitutions $(\rho, 1)P\cdot(0, \rho)=\rho!$ is obtained, the unary factorial function. As examples, it is found that $0!=1$ and $\rho!=\infty$ if ρ is negative. By mathematical induction the following theorem is proved, of which Newton's binomial theorem and Vandermonde's permutation theorem are special cases: If r is a primary number or zero, a, b, d any finite numbers, except that, if r and d are both zero, neither a nor b nor $a+b$ is zero, and m and n are any integers, then $(a+b, d)P(m+n, r)=\sum_{k=1}^{r+1}rC(k-1)\cdot(a, d)P(m, r-(k-1))\cdot(b, d)P(n, k-1)$. By putting $d=0$, $(a+b)^r=\sum_{k=1}^{r+1}rC(k-1)\cdot a^{r-(k-1)}\cdot b^{k-1}$, which is Newton's theorem. By putting $d=1$ and $m=0$ $(a+b)P\rho=\sum_{k=1}^{r+1}rC(k-1)\cdot aP(r-(k-1))\cdot bP(k-1)$, which is Vandermonde's theorem. (Received February 1, 1943.)

105. L. L. Dines: *On linear combinations of quadratic forms.*

The author considers conditions under which m given quadratic forms in n variables admit a linear combination which is (1) definite, or (2) semi-definite. The paper will appear in full in an early issue of Bull. Amer. Math. Soc. (Received December 10, 1942.)

106. H. Schwerdtfeger: *Identities between skew-symmetric matrices.*

Let P, Q be two $2m$-rowed skew-symmetric matrices, P regular. Put $P^{-1}Q=A$. The determinant $|\lambda P-Q|$ equals $\kappa(\lambda)^2$ with $\kappa(\lambda)=k_0\lambda^m-k_1\lambda^{m-1}+\cdots+(-1)^m k_m$ where k_0, k_m are the pfaffian parameters of P, Q, respectively, and $k_1, \cdots, k_{m-1}$ the rational simultaneous invariants of P and Q. By Cayley's identity one has $\kappa(A)^2=(0)$. By means of known theorems (cf. for example, MacDuffee's *Theory of matrices*, Theorems 32.2, 32.3, and 29.3, or A. A. Bennett, Bull. Amer. Math. Soc. vol. 25 (1919) pp. 455–458) it follows that $\kappa(\lambda)$ has as a factor the highest invariant factor $h(\lambda)$ of $\lambda P-Q$, and thus the minimum polynomial of A. Hence follows $h(A)=(0)$ and $\kappa(A)=(0)$. This identity involving the skew-symmetric matrices P, Q is of geometric interest; if $m=2$ one has, for instance: $k_0QP^{-1}Q=k_1Q-k_2P$ whence the elementary theory of a pair of null systems (linear complexes) in projective 3-space can be derived. (Received January 8, 1943.)

Analysis

107. G. E. Albert: *An extension of Korous' inequality for orthonormal polynomials.*

Let $\{q_n(x)\}$ denote the set of polynomials orthonormal on (a, b) with weight functions $p(x)r(x)$ where $0\leqq p(x)$ and $0\leqq r(x)\leqq M$. If a non-negative polynomial $\pi_m(x)$ of degree m can be found such that the quotient $\pi_m(x)/r(x)$ satisfies a Lipschitz condition on (a, b) and if $\{p_n(x)\}$ denotes the set of polynomials orthonormal on (a, b) with weight function $p(x)[\pi_m(x)]^2$ then if the polynomials $\{p_n(x)\}$ are bounded uniformly with respect to n and x on any subset of (a, b) the same is true of the set $\{q_n(x)\}$. This result follows from an inequality that is established by the same procedure as that used on an equiconvergence theorem by L. H. Miller and the author (abstract 49-3-108). If $r(x)$ is bounded from zero and satisfies a Lipschitz condition on (a, b), the inequality mentioned reduces essentially to an inequality due to Korous (G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications vol. 23, 1939, p. 157). (Received January 19, 1943.)

108. G. E. Albert and L. H. Miller: *Equiconvergence of series of orthonormal polynomials.* Preliminary report.

Walsh and Wiener (Journal of Mathematics and Physics vol. 1 (1922)) found necessary and sufficient conditions for the equiconvergence of the expansions of an arbitrary function in terms of different systems of functions orthonormal on a finite interval. In the present paper these conditions are applied to the study of polynomials orthonormal relative to weight functions satisfying a variety of hypotheses. A remarkably simple proof is obtained for an equiconvergence theorem that includes one published by Szegö (*Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publication, vol. 23, 1938, Theorem 13, 1.2) and the results given by Peebles (Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 97–104). The application of the Walsh-Wiener conditions is based upon the observation that if $K_n^{(1)}(x, t)$ and $K_n^{(2)}(x, t)$ are the

respective kernel polynomials for the systems orthonormal with respect to the weight functions $p(t)r_1(t)$ and $p(t)r_2(t)$, then (subject to integrability conditions) for arbitrary fixed x the integral $\int_{-1}^{1} p(t)r_2(t)[r_1(t)r_2^{-1}(t)K_n^{(1)}(x,t)-K_n^{(2)}(x,t)]^2dt$ is the least squares integral of order n for the function $r_1(t)r^{-1}(t)K_n^{(1)}(x,t)$ with respect to the system of weight $p(t)r_2(t)$. An expedient choice of a polynomial of degree n to replace $K_n^{(2)}(x,t)$ and study of the result completes the proof. No asymptotic formulas of any kind are needed. (Received December 9, 1942.)

109. E. F. Beckenbach: *On conjugate harmonic functions.*

According to N. Cioranesco (*Sur les fonctions harmoniques conjuguées*, Bull. Sci. Math. vol. 56 (1932)), a set of n conjugate harmonic functions x_j of n independent variables is a set such that (1) the x_j are harmonic and (2) the function $-\sum(x_j+a_j)^{(2-n)/2}$ is harmonic or $-\infty$ for all values of the parameters a_j. It is now shown that there is a redundancy in the above characterization, for condition (1) is implied by condition (2). The result is extended by means of subharmonic functions to sets of m conjugate harmonic functions of n independent variables, $m \geqq n$. (Received December 31, 1942.)

110. R. H. Cameron and W. T. Martin: *An expression for the solution of a class of nonlinear integral equations.*

The authors give an expression for the solution of the integral equation $\phi(x) = f(x) + \int_0^x F[x, \xi, \phi(\xi)]d\xi$, where $F(x, \xi, y)$ is continuous in $0 \leq x \leq 1$, $0 \leq \xi \leq 1$, $-\infty < y < \infty$ and satisfies the uniform Lipschitz condition $|F(x, \xi, y_2) - F(x, \xi, y_1)| < M|y_2 - y_1|$. For any continuous function $f(x)$ the solution $\phi(x)$ is given as the limit in the mean (in the L_2-sense) of the quotient of two Wiener integrals (averages) over the space of all continuous functions. The proof is carried through for more general nonlinear functional equations which include the above as a special case. (Received January 30, 1943.)

111. M. M. Day: *Uniform convexity.* III.

This paper fills out certain results obtained by the writer in two earlier papers (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317 and pp. 504–507). It also contains the following theorem: If a normed vector space is uniformly convex in the neighborhood of a single point on the unit sphere, then it is isomorphic to a uniformly convex space. (Received January 23, 1943.)

112. George Piranian: *On the convergence of certain partial sums of a Taylor series with gaps.*

Let $f(z)$ be defined by the series $\sum_{n=1}^{\infty} c_n z^{\lambda_n}$ where $\limsup |c_n|^{1/\lambda_n} = 1$, and let $\theta_n = \lambda_{n+1}/\lambda_n - 1$, $M(r) = \max_{|z|=r} |f(z)|$, and $S_n(z) = \sum_{p=1}^{n} c_p z^{\lambda_p}$. If $\limsup \{\log [M(1-\theta_{n_i}^2)/\theta_{n_i}]/\lambda_{n_i}\theta_{n_i}^2\} < \infty$, then $\lim S_{n_i}(z) = f(z)$ at all regular points of $f(z)$ on the circle $|z| = 1$. (Received December 12, 1942.)

113. R. M. Robinson: *Analytic functions in circular rings.*

The fundamental lemma on which this paper depends is the following: *If $f(z)$ is regular and single-valued in the ring $q \leq |z| \leq 1$, except for one simple pole on the negative real axis, and if $|f(z)| \leq 1$ on both boundaries, then $|f(z)| < 1$ for $q < |z| < 1$, that is, on*

the radius opposite the pole. Various applications are given, including the determination of the sharp bound in Hadamard's three circles theorem. That is, we suppose that $f(z)$ is regular and single-valued for $q \leq |z| \leq 1$, that $|f(z)| \leq p$ for $|z| = q$ and $|f(z)| \leq 1$ for $|z| = 1$, and find the largest possible value for $|f(z_0)|$, where z_0 is some point within the ring. A formula for the bound is given in terms of theta functions, and the problem is also discussed geometrically. In particular, if $q < p < 1$, then the maximum value of $|f(z_0)|$ is attained by a function $f(z)$ which is univalent in $q < |z| < 1$, and maps this ring on $|w| < 1$ excluding an arc of $|w| = p$. (Received January 23, 1943.)

114. Raphael Salem: *Sets of uniqueness and sets of multiplicity.*

An algebraic integer α having the property that all its conjugates have their moduli inferior to 1 will be called a "Pisot number" (α is necessarily real and greater than 1). The following theorems are proved: I. Let $0 < \xi < 1$. If the Fourier-Stieltjes transform $\prod_{k=0}^{\infty} \cos \pi u \xi^k$ does not tend to zero for $u \to \infty$, then $1/\xi$ is a Pisot number. II. Let $0 < \xi < 1/2$, and let P be the symmetrical perfect set of Cantor type and of constant ratio of dissection ξ constructed on $(0, 2\pi)$ (relative length of the black intervals is $1-2\xi$). Then P is a set of uniqueness for trigonometrical series if (and only if) $1/\xi$ is a Pisot number. III. There exist Pisot numbers of the form $2+\epsilon$, ϵ being positive and arbitrarily small; hence, there exist sets of uniqueness which are of Hausdorff dimensionality as near to 1 as desired. (Received January 11, 1943.)

115. Gabor Szegö: *On the oscillation of differential transforms.* IV. *Jacobi polynomials.*

Let $\alpha \geq 0$, $\beta \geq 0$, $c \geq 0$. In a recent paper (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 463–497, cf. p. 489), E. Hille proved the following two theorems: (A) The differential operator $\vartheta - c = (1-x^2)D^2 + [\beta - \alpha - (\alpha+\beta+2)x]D - c$, $D = d/dx$, does not diminish the number of the sign changes of a function in $-1 < x < +1$; (B) If the number of the sign changes of $(\vartheta - c)^k f(x)$ remains less than or equal to N for all k, $k = 1, 2, 3, \cdots$, then $f(x)$ is a polynomial of degree less than or equal to N. The purpose of the present note is the extension of Theorem A to $\alpha > -1$, $\beta > -1$ and of Theorem B to arbitrary real values of α and β, in the latter case with the modification that the possible degree of the polynomial $f(x)$ is less than or equal to $N + \gamma$, $\gamma = \gamma(\alpha, \beta, c)$. (Received January 20, 1943.)

Applied Mathematics

116. Stefan Bergman: *A formula for the stream function in compressible fluid flow.*

Using the hodograph method and a general representation for the stream function of a flow of an incompressible fluid (see Bergman, *Hodograph method in the theory of compressible fluid*, Publication of Brown University, 1942) the author gives an explicit formula for the stream functions of flows of certain types. (Received January 27, 1943.)

117. Nathaniel Coburn: *Boundary value problems in plane plasticity.* Preliminary report.

The following problem is discussed in this paper: given an infinite plate of perfectly plastic material bounded by the x-axis; to determine the stresses within the plate when the stresses on the boundary are known. First, the equation of plasticity (yield

condition) is linearized. It is shown that, in this case, the above problem reduces to finding a solution $F(x, y)$ for the hyperbolic equation under given Cauchy data. Next, it is shown that for certain types of boundary conditions, the stresses corresponding to the linear problem are equal to the stresses corresponding to the original nonlinear problem over a series of equally spaced lines parallel to the boundary (x-axis). The situation is analogous to two surfaces which do not coincide but do intersect in curves whose projections on the xy-plane are parallel lines. The method may be modified to give approximate solutions of the nonlinear problem throughout the plate. Some examples are worked. An indication is given of how the method may be applied to the finite rectangular plate. (Received January 2, 1943.)

118. H. J. Greenberg: *Application of a summability method in solving boundary value problems.*

In order to apply the method of particular solutions for solving boundary value problems for the elliptic linear partial differential equation $L(U)=0$, S. Bergman (Duke Math. J. vol. 6 (1940) p. 541) has introduced the complete set $P_{2n-a}(z)$ of such solutions, by means of which every function U, $L(U)=0$, regular in the circle $x^2+y^2<R^2$, $R>0$, can be developed in the series S: $U(z)=\sum a_k P_k(re^{i\phi})$. By studying the "associated function" $\sum a_k\Gamma(1/2)\cdot\Gamma(k+1/2)z^k/\Gamma(k+1)$ necessary and sufficient conditions are given for the series S to be convergent on $|z|=R$ and, what is more important for applications, for S to be $(C, 1)$ summable on this circle of convergence. Conditions are given in each case under which the values thus obtained are the boundary values of the function $U(z)$. Using these results a method is given for the actual solution of boundary value and characteristic value problems for the equation $L(U)=0$. (Received January 29, 1943.)

GEOMETRY

119. John DeCicco: *Extensions of certain dynamical theorems of Halphen and Kasner.*

The theorem of Halphen which characterizes central fields of force and the general theorem of Kasner concerning one-third the curvatures is extended to generalized fields of force in space, which depend upon the position of the point and direction. The number of generalized fields of force whose dynamical trajectories are all plane curves is $\infty^{f(2)+f(4)+2f(5)}$. The ∞^5 generalized trajectories consist of ∞^2 systems of ∞^3 generalized plane trajectories, each system lying in a plane tangent to a given surface Σ. In an arbitrary positional field of force, Kasner showed that the rest trajectory and line of force through a given point O have the same osculating plane and that the ratio ρ of the curvature of the rest trajectory to that of the line of force is 1/3. For generalized fields of force this theorem is no longer valid. All generalized fields of force such that the rest trajectory and the line of force through any point O of the space have the same osculating plane are determined; and, also in this class, the subclass of all generalized fields of force for which $\rho=1/3$. (Received January 2, 1943.)

120. Edward Kasner and John DeCicco: *Generalized dynamical trajectories in space.*

The differential geometry of positional fields of force has been developed in *Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3, 1913. In abstract 48-11-329, the authors began the study of generalized fields

of force in the plane which depend not only upon the position of the point but also upon the direction through the point. In this paper, the geometry of the dynamical trajectories of such generalized fields of force in space is studied. It results that the ∞^5 dynamical trajectories are completely characterized by two geometric properties as follows. For each of the ∞^1 curves of the ∞^5 trajectories which pass through a given lineal element E, construct the osculating plane and sphere at E. The two properties are: (I) The ∞^1 trajectories all have the same osculating plane, and (II) The locus of the centers of the osculating spheres is a straight line. Finally velocity systems are defined and shown to be curvature trajectories. (Received December 28, 1942.)

Statistics and Probability

121. Henry Scheffé: *On solutions of the Behrens-Fisher problem, based on the t-distribution.*

The problem is to find confidence intervals for the difference of the means of two normal populations when the ratio of their variances is unknown. Certain disadvantages inhere in solutions hitherto proposed: Some data are completely discarded when the sample sizes are unequal (Neyman), or the confidence coefficient is not known exactly (Fisher), or existing tables are inadequate if the confidence coefficient is to be 95 or 99 per cent (Wilks). The present solution is as follows: Let the samples be $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_n)$, where the x's and y's are mutually independently normally distributed, the former with mean a_x and variance σ_x^2, the latter with mean a_y and variance σ_y^2. Assume $m \leqq n$. Let $\bar{x}$, $\bar{y}$ be the sample means, and $\delta = a_x - a_y$. Then $(\bar{x} - \bar{y} - \delta)/Q^{1/2}$ has the Fisher t-distribution with $m-1$ degrees of freedom if $m(m-1)Q = \sum_{i=1}^{m}(u_i - \bar{u})^2$, $u_i = x_i - (m/n)^{1/2} y_i$, $\bar{u} = \sum_{i=1}^{m} u_i/m$. This leads immediately to confidence intervals for δ, and it is shown that of a certain class these have the minimum expected length. (Received January 7, 1943.)

Topology

122. M. G. Ettlinger: *On irreducible continuous curves.*

It is proved that, in a connected space satisfying Axioms 0–2 of R. L. Moore's *Foundations of point set theory* (Amer. Math. Soc. Colloquium Publications, vol. 13, 1932), every compact and closed point set is a subset of a compact continuous curve, and every compact and closed point set having no continuum of condensation is a subset of a compact hereditary continuous curve. It is shown that if, in a space satisfying Axioms 0–1, M is a locally compact continuous curve which is an irreducible continuum about a closed subset T of M, then every continuum of condensation of M is a continuum of condensation of T. It is then demonstrated that in a connected space satisfying Axioms 0–2, every compact and closed point set having no continuum of condensation is a subset of a compact continuous curve having no continuum of condensation. (Received January 21, 1943.)

123. O. G. Harrold: *A decomposition theorem for certain compacta.*

Let ψ_n denote the property of being locally connected in dimension n in the sense of homotopy $(n-LC)$ at a point p of a compactum X. Let Δ_n denote the ψ_n-singular points of X. Let $\mathfrak{C}_n^{n+1}$ denote the class of compacta which are LC^n $(i-LC$, $i=0, 1, \cdots, n)$ and such that small singular $(n+1)$-spheres bound. If $X \in \mathfrak{C}_n^{n+1}$, either Δ_{n+1} is vacuous or contains nondegenerate connected sets. Also, analogous to

results of R. L. Wilder (*Decompositions of compact metric spaces*, Amer. J. Math. vol. 63 (1941) pp. 691–697), if $X \in \mathfrak{C}_n^{n+1}$ and the ψ_{n+1}-prime parts of X are simply-i-connected, $i=1, 2, \cdots, n+1$, then the space X' of ψ_{n+1}-prime parts of X is LC^{n+1}. For $n \geq 1$, by use of a well known theorem due to Hurewicz, these theorems may be stated in terms of Vietoris cycles with integer coefficients. (Received January 30, 1943.)

124. A. H. Stone: *Connectedness and coherence.*

This paper is concerned with the incidence geometry of a topological space S, that is, the relations between the numbers of components of sets and their frontiers, unions and intersections, expressed in terms of the zero-dimensional Betti number b_0 and the coherence $c=r+1$ (Eilenberg, Fund. Math. vol. 27 (1936)). Among others, the following theorems are obtained. The coherence of a space may be defined using open sets instead of closed ones. For any set A, $b_0(\mathcal{J}(A)) \leq b_0(\bar{A}) + b_0(\overline{(C(A))}) + c(S) - 1$, and $\sum b_0(\mathcal{J}(C)) \leq b_0(\bar{A}) + c(S) - 1$, the summation extending over the components C of the complement of A. A theorem of Eilenberg on coverings of order 3 is extended and sharpened, yielding the corollary that if S is unicoherent and $\mathcal{J}(A) \cap \mathcal{J}(B) \subset A \cap B \cap \mathfrak{I}(A \cup B)$ then $b_0(A \cap B) + b_0(A \cup B) = b_0(A) + b_0(B)$. Several other theorems, as well as extensions of some known results, are obtained for unicoherent spaces, and some of them are extended to the multicoherent case. Most of the theorems apply to sets which need not be open or closed. S is assumed to be normal; but if S is unicoherent, methods are developed which dispense completely with separation axioms. Applications are made to spaces of coherence 2, and to euclidean space. (Received January 20, 1943.)

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